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1991 J. Phys. A: Math. Gen. 24 L1165

(http://iopscience.iop.org/0305-4470/24/19/008)

J. Phys. A: Math. Gen. 24 (1991) L1165–L1174. Printed in the UK

LETTER TO THE EDITOR

Exactly solvable potentials and the concept of shape invariance

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Received 30 April 1991, in final form 12 August 1991

Abstract. We analyse the relationship between shape invariance and exactly solvable potentials. Using earlier results, we show that all actually known shape invariant potentials with a single parameter can be recovered quite simply. Extension to the case of shape invariant potentials with two parameters as well as to non-shape-invariant but exactly solvable potentials are also possible.

According to Gedenshtein (1983), the concept of shape invariance can be formulated as follows:

$$f_2(a,x)=f_1(a_1,x)+R(a_1) \tag{1}$$

in which f_1, f_2 are the SUSY partners potentials (Witten 1981), a is an ensemble of parameters, $a_1=F(a)$, F is a given transformation operation on these parameters. The following consequences are interesting:

(i) It provides an elementary but exact determination of the energy spectrum

$$E_n=\sum_{i=1}^{n} R(a_i) \tag{2}$$

where n labels the nth excited state, $R(a_i)$ are quantities independent of x, $a_i=F^{(i)}(a)$.

(ii) The nth excited wavefunction $\psi_n^{(-)}(a,x)$ is formally given by (Sukumar 1985, Dutt *et al* 1988)

$$\psi_n^{(-)}(a,x)=N_0\prod_{i=1}^{n-1} A^+(a_i,x)\psi_0(a,x) \tag{3}$$

N_0 is a normalization constant and $A^{\pm}(a_i,x)$ represents the ith 'ladder operator'

$$A^{\pm}(a_i,x)=\pm\frac{\mathrm{d}}{\mathrm{d}x}+v'(a_i,x) \tag{4}$$

with $v'(a_i,x)=\mathrm{d}v/\mathrm{d}x$, $v(a_i,x)$ is the corresponding superpotential while ψ_0 is the ground state wavefunction ($E_0=0$) which can be determined from $v'(a,x)$

$$\frac{\psi_0'(a,x)}{\psi_0(a,x)}=v'(a,x). \tag{5}$$

The recent revival of an old problem of quantum mechanics dealing with exactly solvable potentials, on the other hand (Schrödinger 1941, Infeld and Hull 1951, Bhattacharjie 1962 and others) with its close relation with the concept of shape invariance (Dabrowska *et al* 1987, Dutt *et al* 1988, Levai 1989, Cooper *et al* 1989 and others) have in fact opened new perspectives particularly in the construction of this type of potential.

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0305-4470/91/191165+10$03.50

In an attempt to understand more on this subject, we were led recently to raise the following question: is the concept of shape invariance (in the sense of Gedenshtein) not only a sufficient but also a necessary condition in generating (although with some possible exceptions) exactly solvable potentials? Obviously this is related to another aspect which was discussed earlier (Cao 1990b) where we wondered whether the theorem on the existence of a superpotential which includes all actually known shape-invariant potentials may also incorporate some special types of non-shape-invariant but exactly solvable potentials or, in other words, how the concept of shape invariance can be taken into account by this theorem.

The clarification of this question will constitute the main objective of the present work in which we first show how shape-invariant potentials with one parameter can be constructed.

Next, the technique of $\hat{C}$ transformation will be briefly reviewed and will enable us to generate shape invariant potentials with two parameters. The discussion is extended to the case of multiparameters particularly in the construction of non-shape-invariant but exactly solvable potentials.

'Existence theorem' and the shape invariance concept. We are concerned with a system of two second-order differential equations

$$\left[\frac{d^2}{dx^2}+f_i(x)\right]y_i=2Ey_i \qquad i=1,2.$$

The theorem states that the two solutions y_1, y_2 are SUSY partners if $f_1(x)$, $f_2(x)$ are related by the following conditions:

$$(f_1+f_2)^{1/2}=\frac{1}{2^{1/2}}\int^x (f_1-f_2)\,dx \tag{6}$$

from which we may extract some immediate consequences.

Statement 1. For any arbitrary analytic function $f_1(x)$, (6) implies that it is always possible to construct its SUSY partner $f_2(x)$; they are both exactly solvable if one of them is.

Proof. Let:

$$r=\tfrac{1}{2}(f_1+f_2) \qquad s=\tfrac{1}{2}\int^x (f_1-f_2)\,dx \qquad \text{so that} \qquad \begin{aligned} f_1&=r+s' \\ f_2&=r-s'. \end{aligned} \tag{7}$$

From (6) we have

$$r=s^2 \tag{8}$$

and s is determined by the Ricatti equation

$$s'+s^2-f_1(x)=0. \tag{9}$$

Its solution is

$$s=\frac{d}{dx}\log y_{1,0}(x) \tag{10}$$

where $y_{1,0}(x)$ is the solution of the Schrödinger equation corresponding to the zero energy ground state. In the SUSY terminology we recognize that s can be identified with v' so that

$$f_1 = v'^2 - v'' \qquad f_2 = v'^2 + v''. \tag{11}$$

The 'ladder operator' $A^{\pm}$ are defined by $A^{\pm} = \pm\,\mathrm{d}/\mathrm{d}x + v'$ and the couple (y_1, y_2) must satisfy the system:

$$A^+ y_1 = (2E)^{1/2} y_2 \qquad A^- y_2 = (2E)^{1/2} y_1. \tag{12}$$

From (1) and (11) we have:

$$f_1(a_1, x) + R(a_1) = f_1(a, x) + 2v''(a, x). \tag{13}$$

Note that if $a \equiv a_1$ (no change of the parameters) the quantity $v'(a, x)$ must be constant. If $a \neq a_1$, the simplest structure of $f_1(a, x)$ may be taken as:

$$f_1(a, x) = h(a, x) + C(a) \tag{14}$$

in which $C(a)$ is a constant and $h(a, x)$ an analytic function of x. Consequently they must satisfy simultaneously the two conditions:

$$\begin{aligned} &h(a_1, x) - h(a, x) = 2v''(a, x) \\ &C(a) - C(a_1) = R(a_1). \end{aligned} \tag{15}$$

Which can be summarized in a second statement.

Statement 2. The function $f_1(a, x)$ is shape invariant if it has the structure (14) in which the quantities $h(a, x)$, $C(a)$ must satisfy simultaneously conditions (15).

From these two statements, it will be easy to check the following relations which are essential in our construction of shape-invariant potentials

$$v' = \frac{1}{2}\,\frac{h'(a_1, x) + h'(a, x)}{h(a_1, x) - h(a, x)} \tag{16}$$

$$\frac{1}{2}\left[\frac{h(a_1, x) + h(a, x)}{h(a_1, x) - h(a, x)}\right]^2 - [h(a_1, x) + h(a, x)] = m = \text{constant} \tag{17}$$

$$C(a) = \tfrac{1}{2}m \tag{18}$$

with $h' = \mathrm{d}h/\mathrm{d}x$; (16) defines the superpotential $v(a, x)$, (17) is a differential equation from which $h(a, x)$ can be inferred if the constant m is chosen appropriately and (18) leads to the determination of the energy spectrum from (2).

Shape invariant potentials with one parameter. We consider now a few simple analytical forms of $h(a, x)$ such that (17) can be solved exactly.

Example 1. Let $h(a, x) = \varepsilon A(a)$, $g(x)$ (separability) in which $A(a)$, $g(x)$ are any function of a and x, $\varepsilon = \pm 1$. From (17) we have

$$\frac{1}{2}\left[\frac{A(a_1) + A(a)}{A(a_1) - A(a)}\right]^2 \left(\frac{g'}{g}\right)^2 - [A(a_1) + A(a)]g = m.$$

The analytical form of the solution will depend on ε and on the sign of the constant m. For instance if we take

$$\varepsilon=+1 \qquad m \gtrless 0 \qquad |m/2| = \left[\frac{A(a_1)+A(a)}{A(a_1)-A(a)}\right]^2$$

$$a_1 = a-1 \qquad C(a) = -a^2 \qquad A(a) = a(a+1) \quad .$$

then (16), (17) yield the following result:

$$v'(a,x) = \begin{cases} a \coth x \\ -a \tan x. \end{cases}$$

If $\varepsilon=-1$, $m \gtrless 0$ then

$$v'(a,x) = \begin{cases} a \tanh x \\ a \cot x. \end{cases}$$

Finally, if we take $m=0$ then $v'(a,x)=a/x$ with

$$A(a)=a(a+1) \qquad a_1=a-1 \qquad C(a)=0.$$

Summarizing, we see that with the five different forms of the superpotential v' given above, relation (11) will lead to the well known shape-invariant potentials reported in most actual compilations (Dutt *et al* 1987, Dabrowska *et al* 1987, Levai 1989).

Example 2. Let $h(a,x)=g^2+2ag$ where g and a have the same meaning as above. The equation (17) can also be solved exactly if we take $m=-2(a+\frac{1}{2})^2$; for an arbitrary constant of integration B we find $g=B\,\mathrm{e}^{-x}$ and $v'(a,x)=B\,\mathrm{e}^{-x}-(a+\frac{1}{2})$, $C(a)=(a+\frac{1}{2})^2$ and

$$f=B^2\,\mathrm{e}^{-2x}+2B(a+1)\,\mathrm{e}^{-x}+(a+\tfrac{1}{2})^2$$

which is the well known generalized Morse potential.

Example 3. Note also for completeness that, in the case $a_1 \equiv a$ (no change of parameters), v' must be a linear function of x, so in setting $v'(x)=\frac{1}{2}\omega x$, we have the harmonic oscillator problem which is not strictly shape invariant but, however, an exactly solvable potential.

We shall now use these results to extend the analysis to the case of shape-invariant potentials with two parameters through the technique of the $\hat{C}$ transformation which will first be briefly described.

The C transformation. Keeping the same notation as in a previous paper we consider a transformation such that (Cao 1991)

$$\phi=\hat{C}y \tag{19}$$

in which $y=(y_1, y_2)$ are the SUSY partners defined in (12), $\phi=(\phi_1,\phi_2)$ and $\hat{C}=c(x)I$, I is the unit matrix and $c(x)$ is for the moment an arbitrary function of x. With this transformation the new 'ladder operators' $\hat{A}^{\pm}$ are

$$\hat{A}^{+}_{\alpha}=\frac{\mathrm{d}}{\mathrm{d}x}+\hat{v}'_{\alpha} \qquad \hat{A}^{-}_{\beta}=-\frac{\mathrm{d}}{\mathrm{d}x}+\hat{v}'_{\beta} \qquad \hat{v}'_{\alpha}=v'+\frac{c'}{c} \qquad \hat{v}'_{\beta}=v'-\frac{c'}{c} \tag{20}$$

where v' is the superpotential taken from examples 1, 2, 3.

The $\hat{C}$ transformation yields the new system of equations

$$\hat{A}^{+}_{\alpha}\hat{\phi}_1=(2E)^{1/2}\hat{\phi}_2 \qquad \hat{A}^{-}_{\beta}\hat{\phi}_2=(2E)^{1/2}\hat{\phi}_1 \tag{21}$$

in which obviously the components $\hat{\phi}_1, \hat{\phi}_2$ are not SUSY partners since $\hat{A}_\alpha^+, \hat{A}_\beta^-$ are not adjoint. We may nevertheless point out the following interesting properties.

(i) The commutativity relation is invariant in this transformation

$$[\hat{A}_\beta^-, \hat{A}_\alpha^+] = [A^-, A^+] = -2v''.$$

(ii) Defining the new opeator '$\hat{Q}$' as

$$\hat{Q}_\alpha^+ = \begin{pmatrix} 0 & \hat{A}_\alpha^+ \\ 0 & 0 \end{pmatrix} \qquad \hat{Q}_\beta^- = \begin{pmatrix} 0 & 0 \\ \hat{A}_\beta^- & 0 \end{pmatrix}$$

and the 'Hamiltonians' $\hat{H}$ by

$$\hat{H} = \{\hat{Q}_\alpha^+, \hat{Q}_\beta^-\}$$

it can be verified that

$$[\hat{Q}_\alpha^+, \hat{H}] = [\hat{Q}_\beta^-, \hat{H}] = 0.$$

The representation $\hat{\phi}_1, \hat{\phi}_2$ may become useful in certain cases, for instance when we use the coordinate transformation $x \to r$ such that $f^{2m}(x) = \mathrm{d}r/\mathrm{d}x$ and choosing $c(x) = f^m(x)$ (m is a parameter) then it can be shown that the present method leads to the 'f operator transformation' approach already discussed by other authors (Cooper *et al* 1989) who showed that, in certain cases, it can generate the Natanzon potentials (i.e. potentials with solutions expressed by a combination of hypergeometric functions) which are not shape invariant but exactly solvable under special conditions (see also Cervero 1991, Levai 1991).

In the present work we shall however follow a different path by considering the couples $(\hat{\phi}_1, \hat{\phi}_{1,s})$; $(\hat{\phi}_2, \hat{\phi}_{2,s})$ defined by:

$$\begin{aligned} &\hat{A}_\alpha^+\hat{\phi}_1 = (2E)^{1/2}\hat{\phi}_{1,s} \qquad \hat{A}_\alpha^-\hat{\phi}_{1,s} = (2E)^{1/2}\hat{\phi}_1 \\ &\hat{A}_\beta^+\hat{\phi}_{2,s} = (2E)^{1/2}\hat{\phi}_1 \qquad \hat{A}_\beta^-\hat{\phi}_2 = (2E)^{1/2}\hat{\phi}_{2,s} \\ &\hat{A}_{\alpha,\beta}^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}x} + v'_{\alpha,\beta}(x). \end{aligned} \tag{22}$$

By construction, $(\hat{\phi}_{1,s}, \hat{\phi}_1)$, $(\hat{\phi}_2, \hat{\phi}_{2,s})$ are SUSY partners and as the treatment for the cases α and β are quite similar, from now on we shall omit the indices for simplicity. For example the equations for $\hat{\phi}_1$ and $\hat{\phi}_{1,s}$ are

$$\hat{A}^-\hat{A}^+\hat{\phi}_1 = 2E\hat{\phi}_1 \qquad \hat{A}^+\hat{A}^-\hat{\phi}_{1,s} = 2E\hat{\phi}_{1,s} \tag{23}$$

or, more explicitly,

$$\begin{aligned} &\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v'^2 - v'' + 2v'\left(\frac{c'}{c}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c'}{c}\right) + \left(\frac{c'}{c}\right)^2\right]\hat{\phi}_1 = 2E\hat{\phi}_1 \\ &\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v'^2 + v'' + 2v'\left(\frac{c'}{c}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c'}{c}\right) + \left(\frac{c'}{c}\right)^2\right]\hat{\phi}_{1,s} = 2E\hat{\phi}_{1,s}. \end{aligned} \tag{24}$$

As $c(x)$ still remains arbitrary, it will serve to incorporate the second parameter in the new superpotential $v(x)$ defined by:

$$\hat{v}'(x) = v'(x) + \frac{c'}{c}. \tag{25}$$

Remembering that the couple y_1, y_2 are in fact solutions of shape-invariant potentials corresponding to v', the solution $\hat{\phi}_1$ of (24) can be written as

$$\hat{\phi}_1 = c(x) y_1 \tag{26}$$

if $c(x)$ is chosen appropriately.

However, the determination of the energy spectrum of system can be done analytically only under certain conditions, the concept of shape invariance being one of these conditions. In spite of these limitations, one can nevertheless foresee that the present approach is susceptible to generate a wide variety of exactly solvable or solvable potentials if normalization requirements on $\hat{\phi}_1$ can be met (by solvable potential we simply mean here that the solution of the Schrödinger equation can be obtained analytically from (26) but the energy spectrum $\{E_n\}$ cannot be determined by elementary means such as in (2)).

Shape invariant potentials with two parameters. Let

$$\hat{f}_{1,s}(x) = f_{1,s}(x) + 2v'\left(\frac{c'}{c}\right) \mp \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c'}{c}\right) + \left(\frac{c'}{c}\right)^2 \tag{27}$$

in which $f_{1,s}(x)$ is given by (11) and is assumed to be shape invariant so that the problem now is how to choose $c(x)$ such that $\hat{f}_{1,s}(x)$ is also shape invariant. We have noted three interesting cases which are:

(i) $$2v'\left(\frac{c'}{c}\right) = k \qquad k \text{ constant} \tag{28}$$

(ii) $$2Kv'\left(\frac{c'}{c}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c'}{c}\right) \qquad K \text{ constant} \tag{29}$$

(iii) $$2v'\left(\frac{c'}{c}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{c'}{c}\right) + \left(\frac{c'}{c}\right)^2 = 0. \tag{30}$$

Note that the last case concerns the problem of isospectral potentials which was discussed previously (Cao 1991) so that below we shall mainly be concerned with the first two cases.

Example 4. We shall start with the simple Pôsch–Teller potential $v' = a \tanh x$, taking a to be the first parameter and solve, for instance, equation (28). Obviously the solutions of this equation will depend on the parameter k which is arbitrary. We may then consider the cases $k \propto v'$, $k > 0$, $k < 0$ and set $k = 2ab$, b constant. We find the following forms of $\hat{v}'(k)$ and $\hat{f}_1^{(k)} = \hat{f}^{(k)}$

$$\hat{v}' = a \tanh x + \frac{b}{a} \qquad \hat{f}^{(0)} = a^2 + \frac{b^2}{a^2} - \frac{a(a+1)}{\cosh^2 xc} + 2b \tanh x \tag{31}$$

$$\hat{v}' = a \tanh x - b \coth x \qquad \hat{f} = (a-b)^2 - \frac{a(a+1)}{\cosh^2 x} + \frac{b(b-1)}{\sinh^2 x} \tag{32}$$

$$E_n = (a-b)^2 - (a-b-2n)^2$$

$$\hat{v}' = a \tanh x + b \coth x \qquad \hat{f} = (a+b)^2 - \frac{a(a+1)}{\cosh^2 x} + \frac{b(b+1)}{\sinh^2 x} \tag{33*}$$

$$E_n = (a+b)^2 - (a+b-2n)^2.$$

If we start with $v' = a \coth x - \bar{b}/a$ we have

$$\hat{f}^{(0)} = a^2 + \frac{b^2}{a^2} + \frac{a(a+1)}{\sinh^2 x} + 2b \coth x$$
$$E_n = a^2 - (a-n)^2 + b^2\left(\frac{1}{a^2} - \frac{1}{(a-n)^2}\right). \tag{34}$$

Remarks.

(i) Equations (31) and (34) are shape invariant with one parameter a; b unchanged.

(ii) Generalizing the above formulation, we may write (note also the validity of (17))

$$f(a, b; x) = h(a, x) + j(b, x) + C(a, b)$$
$$R(a, b) = C(a, b) - C(a_1, b_1).$$

(iii) The notation * means simply that the corresponding result is 'missing' in actual compilations.

In the same manner, if we consider equation (29), the quantities $\hat{v}'$ and $\hat{f}$ will depend on the choice of the constant K which may be zero, positive or negative. The case $K=0$ does not bring anything new but the case $K \neq 0$ is meaningful (b being a constant of integration and we assume $0 < a < b$)

$$\hat{v}' = a \tanh x - \frac{b}{\cosh x} \qquad \hat{f} = a^2 - \frac{a(a+1)+b^2}{\cosh^2 x} - (2a+1)b\frac{\sinh x}{\cosh^2 x}$$
$$E_n = a^2 - (a-n)^2 \tag{35}$$

$$\hat{v}' = a \tanh x + b \cosh x \qquad \hat{f} = a^2 - \frac{a(a+1)}{\cosh^2 x} + b^2 \cosh^2 x - (2a-1)b \sinh x \tag{36}$$

$$\hat{v}' = a \coth x - \frac{b}{|\sinh x|} \qquad \hat{f} = a^2 + \frac{a(a+1)}{\sinh^2 x} - (2a+1)b\frac{\cosh x}{\sinh^2 x} \tag{37}$$

$$\hat{v}' = a \coth x - b|\sinh x| \qquad \hat{f} = a^2 + \frac{a(a+1)}{\sinh^2 x} + b^2 \sinh^2 x - (2a+1)b \cosh x. \tag{38}$$

The solutions are given by (26) in which y are essentially functions of type A in the classification scheme of Infeld and Hull (1953) (Pösch–Teller hypergeometric) while $c(x)$ can be inferred from equations (28), (29) which are elementary in the above cases. Normalization conditions, on the other hand may, in some cases, lead to further restrictions in the use of the parameters.

If we start with $v' = a \tan x$ or $\cot x$ of example 1 and proceed exactly as above in solving successively equations (28), (29), we obtain other types of potentials which are shape invariant with one or two parameters; most of them in fact are already reported in these compilations. We quote here only two missing shape-invariant potentials, which are

$$\hat{v}' = a \tan x + \frac{b}{a} \qquad \hat{f} = -a^2 + \frac{b^2}{a^2} + \frac{a(a-1)}{\cos^2 x} + 2b \tan x \tag{39*}$$

$$\hat{v}' = a \tan x + \frac{b}{\cos x} \qquad \hat{f} = -a^2 + \frac{b^2 + a(a-1)}{\cos^2 x} + (2a-1)b\frac{\sin x}{\cos^2 x}. \tag{40*}$$

Example 5. We may continue with $v' = ax$ or $v' = a/x$ and solve successively equations (28), (29) in order to obtain other types of potentials reported in these compilations. For instance in solving (29) with $k=0$ we recover the well known potential with a Coulomb term widely discussed elsewhere in the literature.

We present here only two simple cases which will be interesting in later developments.

(i) $v' = -b/x$; solving (28) with $k \neq 0$ and with a slight change in notation we find $c(x) = \mathrm{e}^{-\frac{1}{4}\omega x^2}$; $\hat{\phi}(x) = N_0 x^{l+1}\, \mathrm{e}^{-\frac{1}{4}\omega x^2}$ from (26). This result can also be checked by direct substitution in the Schrödinger equation with

$$\hat{f} = \frac{l(l+1)}{x^2} + \tfrac{1}{4}\omega^2 x^2 - \omega(l - \tfrac{1}{2}). \tag{41}$$

(ii) If now $v' = ax$, and solving (29), we have

$$\hat{v}' = ax + b\, \mathrm{e}^{-\frac{1}{2}x^2} \qquad \hat{f} = a^2x^2 + b^2\, \mathrm{e}^{-x^2} + (2a+1)bx\, \mathrm{e}^{-\frac{1}{2}x^2} + a \tag{42}$$

which constitutes an example of a non-shape-invariant potential, but a solvable one. Its solution is

$$\hat{\phi}(x) = N_0 \exp\left[-\tfrac{1}{4}\omega x^2 + \left(\frac{\pi}{2}\right)^{1/2} b\Theta(x/\sqrt{2})\right]$$

where $\Theta(x)$ is the error function defined by:

$$\Theta(x) = \frac{2}{\sqrt{\pi}} \int^x \mathrm{e}^{-t^2}\, \mathrm{d}t.$$

Generalization. The technique of the $\hat{C}$ transformation may be repeated again and again, introducing at each step a new parameter a_i. For instance after the nth step, the wavefunction is

$$\hat{\phi}^{(n)}(x) = \prod_{i=0}^{n-1} c(a_i, x)\phi_0(x)$$

in which $\phi_0(x)$ is the solution corresponding to the starting superpotential $v(x)$. Associated with this wavefunction is the nth superpotential $v^{(n)}(x)$ defined by

$$\hat{v}'^{(n)}(a_n, x) = v' + \sum_{i=1}^{n-1} \frac{c_i'}{c_i} \tag{43}$$

with

$$\hat{f}^{(n)}(a_0, a_1, \ldots, a_n; x) = (\hat{v}'^{(n)})^2 - \hat{v}''^{(n)}. \tag{44}$$

(i) For shape invariant case, this $v^{(n)}(x)$ must be related closely to the preceding ones v, $v^{(1)}, \ldots, v^{(n-1)}$ through equations of the type (28) or (29). This question is actually under investigation.

(ii) If however we are not interested in the shape-invariance problem, then the quantities c_i'/c_i can be chosen almost arbitrarily. In some cases, this may lead to non-shape-invariant but exactly solvable potentials as can be seen in the following example.

Example 6. Let

$$v'(x)=\frac{a_0}{x}-a_2x$$

and consider a function $c(x)$ such that $-c_2'/c_2=Ax^2+B$. Therefore the corresponding effective potential V_{eff} will have the polynomial form:

$$V_{\text{eff}}=b_4x^4+b_3x^3+b_2x^2+b_1x+\frac{\alpha}{x}+\frac{a_0(a_0+1)}{x^2} \tag{45}$$

in which

$$b_4=A^2 \qquad b_3=2Aa_2 \qquad b_2=a_2^2+2AB$$

$$b_1=2[A(a_0-1)+a_2B] \qquad \alpha=2Ba_0.$$

The wavefunction and the energy (eigenvalue) are given by:

$$\begin{aligned}\hat{\phi}(x)&=N_0x^{a_0+1}\exp\{-[Bx+a_2^2x^2+\tfrac{1}{3}Ax^3]\}\\ E_{a_0'}&=2(a_0-1)+B^2.\end{aligned} \tag{46}$$

With some minor modifications in the notation, it can be verified that these results are similar to those obtained in other references (Varshini *et al* 1989, Sutra 1987).

(iii) For the special case of polynomial potentials, it can be noted that the function $c_i(x)\simeq\exp[-a_ix^i]$ so that from (26) we may write

$$\hat{\phi}=\phi_0\exp\left(-\sum_i a_ix^i\right).$$

This remark justifies in a sense the use of the ansatz of Flessas (Flessas 1979, Flessas *et al* 1981) convenient for dealing with this kind of potential.

In combining the theorem on the existence of the superpotential with the concept of shape invariance, it has been possible first to construct shape-invariant potentials with one parameter. Then, with the use of the '$\hat{C}$' transformation, it has been extended to shape-invariant potentials with two parameters including furthermore a few 'missing' ones in actual compilations. This leads us to the following conclusions.

(i) The concept of shape invariance is a sufficient but not necessary condition in the construction of exactly solvable potentials.

(ii) It is effectively possible to construct exactly solvable but non-shape-invariant potentials (example 6).

(iii) It can be noted that all the various types of potential encountered in the present work, namely

the shape-invariant potentials

the non-shape-invariant but exactly solvable potentials

the solvable potentials

can be constructed and incorporated in the frame of the theorem of existence of the superpotential.

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